On Dual Formulation of Gravity. II.
Metric and Affine Connection.

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Abstract

In this note we construct a dual formulation of gravity where the main dynamical object is affine connection. We start with the well known first order Palatini formulation but in (Anti) de Sitter space instead of flat Minkowski space as a background. The final result obtained by solving equations for the metric is the Lagrangian written by Eddington in his book in 1924. Also there is an interesting connection with attempts to construct gravitational analog of Born-Infeld electrodynamics.

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In general, by dual formulation we mean any situation where the very same particle is described by different tensor fields. The most simple and straightforward way to obtain such dual formulation based on the use of first order "parent" Lagrangians. As is well known in flat Minkowski space such dualization procedure leads to different results for massive and massless particles. At the same time in (Anti) de Sitter space-time gauge invariance requires introduction quadratic mass-like terms into the Lagrangians. As a result dualization for massless particles in (Anti) de Sitter spaces [1] goes exactly in the same way as that for massive particles. As an example, we have recently shown [2] that by using well known tetrad formalism it is possible to obtain dual formulation of gravity with the Lorentz connection being the main dynamical field while tetrad is just auxiliary fields which could be expressed in terms of Lorentz connection and its derivatives. But there exist another well known first order formalism for gravity usually called Palatini formalism, the main components being the metric and affine connection. Such formalism differs drastically from the tetrad one because affine connection is not a gauge invariant object (or, geometrically, it is not a covariant tensor) and does not have its own gauge invariance. In spite of this difference, as we are going to show in this note, it is also possible to apply the same dualization procedure to obtain a formulation of gravity where the main dynamical field is the affine connection. Rather naturally and at the same time surprisingly the final result is nothing else but the Lagrangian written by Eddington in 1924 [3]!

Let us start with the first order Lagrangian describing free massless spin-2 particle in flat Minkowski space:

$$\mathcal{L}_0 = h^{\mu\nu}(\partial_\alpha\Gamma_{\mu\nu}{}^\alpha - \partial_\mu\Gamma_\nu) + \eta^{\mu\nu}(\Gamma_{\mu\nu}{}^\alpha\Gamma_\alpha - \Gamma_{\mu\alpha}{}^\beta\Gamma_{\nu\beta}{}^\alpha) \tag{1}$$

Here $h_{\mu\nu}$ is symmetric second rank tensor while $\Gamma_{\mu\nu}{}^\alpha$ is assumed to be symmetric on the lower pair of indices. We denote $\Gamma_\alpha = \Gamma_{\alpha\beta}{}^\beta$, $\Gamma^\alpha = \eta^{\mu\nu}\Gamma_{\mu\nu}{}^\alpha$ (note, that Γ_α and Γ^α are in general different objects). This Lagrangian is invariant under the following local gauge transformations:

$$\delta h_{\mu\nu} = \partial_\mu\xi_\nu + \partial_\nu\xi_\mu - \eta_{\mu\nu}(\partial\xi), \qquad \delta\Gamma_{\mu\nu}{}^\alpha = -\partial_\mu\partial_\nu\xi^\alpha \tag{2}$$

As is well known, if one solves the algebraic equation of motion for the Γ field and put the result back into the Lagrangian one obtains usual second order Lagrangian for the symmetric tensor $h_{\mu\nu}$. In order to have a possibility to construct dual formulation where the main dynamical object is Γ we move from the flat Minkowski space to (Anti) de Sitter space. Let $\bar{g}_{\mu\nu}$ be a metric for this space (it is not a dynamical quantity, just a background field here) and D_μ — derivatives covariant with respect to background connection winch is torsionless and metric compatible:

$$D_\alpha\bar{g}_{\mu\nu} = 0, \qquad [D_\mu, D_\nu]v_\alpha = \bar{R}_{\mu\nu,\alpha}{}^\beta(\bar{g})v_\beta = \kappa(\bar{g}_{\mu\alpha}\delta_\nu{}^\beta - \delta_\mu{}^\beta\bar{g}_{\nu\alpha})v_\beta \tag{3}$$

where $\kappa = -2\Lambda/(d-1)(d-2)$. First of all we have to replace in the Lagrangian as well as in the gauge transformations the flat metric $\eta_{\mu\nu}$ by $\bar{g}_{\mu\nu}$ and partial derivatives ∂_μ by covariant ones D_μ:

$$\begin{aligned}
\mathcal{L}_0 &= h^{\mu\nu}(D_\alpha\Gamma_{\mu\nu}{}^\alpha - D_\mu\Gamma_\nu) + \bar{g}^{\mu\nu}(\Gamma_{\mu\nu}{}^\alpha\Gamma_\alpha - \Gamma_{\mu\alpha}{}^\beta\Gamma_{\nu\beta}{}^\alpha) \\
\delta h_{\mu\nu} &= D_\mu\xi_\nu + D_\nu\xi_\mu - \bar{g}_{\mu\nu}(D\xi), \qquad \delta\Gamma_{\mu\nu}{}^\alpha = -\frac{1}{2}(D_\mu D_\nu + D_\nu D_\mu)\xi^\alpha
\end{aligned} \tag{4}$$

Now (just because covariant derivatives do not commute) our Lagrangian is not invariant under the gauge transformations. Indeed, simple calculations give:

$$\delta\mathcal{L}_0 = \kappa[(d-2)\Gamma_\mu\xi^\mu - \frac{d-3}{2}\Gamma^\mu\xi_\mu - \frac{3d-1}{2}h^{\mu\nu}D_\mu\xi_\nu + h(D\xi)]$$

But gauge invariance could be easily restored by adding terms quadratic in $h_{\mu\nu}$ field to the Lagrangian as well as appropriate corrections for the gauge transformations:

$$\Delta\mathcal{L}_0 = \frac{\kappa(d-1)}{2}[h^{\mu\nu}h_{\mu\nu} - \frac{1}{d-2}h^2]$$

$$\delta'\Gamma_{\mu\nu}{}^\alpha = \frac{\kappa}{2}(\delta_\mu{}^\alpha\xi_\nu + \delta_\nu{}^\alpha\xi_\mu) - \kappa\bar{g}_{\mu\nu}\xi^\alpha \tag{5}$$

Now one can easily solve the equations for the $h_{\mu\nu}$ field, which are also algebraic now, to obtain:

$$h_{\mu\nu} = \frac{1}{\kappa(d-1)}[R_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R] \tag{6}$$

where we introduced a symmetric second rank tensor (it is not a full Ricci tensor yet, only the first part of it):

$$R_{(\mu\nu)} = \frac{1}{2}(D_\mu\Gamma_\nu + D_\nu\Gamma_\mu) - D_\alpha\Gamma_{\mu\nu}{}^\alpha \tag{7}$$

Then if we put this expression back into the initial first order Lagrangian we obtain dual second order formulation for massless spin-2 particle in terms of Γ field:

$$\mathcal{L}_{II} = -\frac{1}{\kappa(d-1)}[R^{\mu\nu}R_{\mu\nu} - \frac{1}{2}R^2] + \bar{g}^{\mu\nu}(\Gamma_{\mu\nu}{}^\alpha\Gamma_\alpha - \Gamma_{\mu\alpha}{}^\beta\Gamma_{\nu\beta}{}^\alpha) \tag{8}$$

A natural question arises: our field $\Gamma_{\mu\nu}{}^\alpha$ has a lot of independent components (40 in $d = 4$ instead of two helicities for massless spin-2 particle), so there should exist a large gauge symmetry in such a model. And indeed, it is easy to check that the kinetic terms in our second order Lagrangian are invariant under the local "affine" transformations:

$$\delta\Gamma_{\mu\nu}{}^\alpha = \partial_\mu z_\nu{}^\alpha + \partial_\nu z_\mu{}^\alpha + \frac{1}{d-1}[\delta_\mu{}^\alpha(\partial z)_\nu + \delta_\nu{}^\alpha(\partial z)_\mu] -$$
$$-\frac{1}{d-1}[\delta_\mu{}^\alpha\partial_\nu z + \delta_\nu{}^\alpha\partial_\mu z] \tag{9}$$

where $z_\mu{}^\nu$ is arbitrary second rank tensor and $z = z_\mu{}^\mu$.

Now, having in our disposal an alternative description for massless spin-2 particle, it is natural to see how an interaction in such dual theory looks like. Nice feature of Palatini formulation is that switching on an interaction is a simple one step procedure [4]. But as we have seen, it is very important for the possibility to construct dual formulations to work not in a flat Minkowski space but in (Anti) de Sitter space. So we start with the usual Lagrangian with the cosmological term:

$$\mathcal{L} = \sqrt{-g}g^{\mu\nu}R_{\mu\nu} + \Lambda\sqrt{-g} \tag{10}$$

where now

$$R_{\mu\nu} = \frac{1}{2}(D_\mu\Gamma_\nu + D_\nu\Gamma_\mu) - D_\alpha\Gamma_{\mu\nu}{}^\alpha + \Gamma_{\mu\nu}{}^\alpha\Gamma_\alpha - \Gamma_{\mu\alpha}{}^\beta\Gamma_{\nu\beta}{}^\alpha \tag{11}$$

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Then we introduce a convenient combination $\hat{g}^{\mu\nu} = \sqrt{-g}g^{\mu\nu}$ and rewrite a Lagrangian as:

$$\mathcal{L} = \hat{g}^{\nu\nu}R_{\mu\nu} + \Lambda det(\hat{g}^{\mu\nu})^{\frac{1}{d-2}} \tag{12}$$

The crucial point here is that first term contains $\hat{g}$ only linearly. As a result it is possible to get complete nonlinear solution of the $\hat{g}$ equations. We obtain (up to some numerical coefficients):

$$\hat{g}^{\mu\nu} \simeq \sqrt{det(R_{\mu\nu})}(R^{\mu\nu})^{-1} \tag{13}$$

At last, if we put this expression back into the first order Lagrangian we obtain (again up to normalization) a very simple and elegant Lagrangian:

$$\mathcal{L} = \sqrt{det(R_{\mu\nu})} \tag{14}$$

And it is just a Lagrangian written by Eddington eighty years ago in his book [3]! This result is very natural because this Lagrangian is the only invariant that could be constructed out of the affine connection alone, without any use of metric or any other objects, but it is exiting that this Lagrangian turns out to be dual formulation of usual gravity theory. Let us stress once again that working in a flat Minkowski space it is very hard if at all possible to give any reasonable physical interpretation to such model. But let us consider this model on a (Anti) de Sitter background. For that purpose we represent a total connection as $\Gamma_{\mu\nu}{}^{\alpha} = \bar{\Gamma}_{\mu\nu}{}^{\alpha} + \tilde{\Gamma}_{\mu\nu}{}^{\alpha}$ where $\bar{\Gamma}_{\mu\nu}{}^{\alpha}$ is a background connection while $\tilde{\Gamma}_{\mu\nu}{}^{\alpha}$ — small perturbation around it (see e.g. [5, 6]). Then for the curvature tensor we will have:

$$R_{\mu\nu,\alpha}{}^{\beta} = \bar{R}_{\mu\nu,\alpha}{}^{\beta} + [D_{\mu}\tilde{\Gamma}_{\nu\alpha}{}^{\beta} + \tilde{\Gamma}_{\mu\alpha}{}^{\rho}\tilde{\Gamma}_{\rho\nu}{}^{\beta} - (\mu \leftrightarrow \nu)] \tag{15}$$

where $\bar{R}_{\mu\nu,\alpha}{}^{\beta}$ is a curvature tensor for the background connection and D_{μ} is a derivative covariant with respect to $\bar{\Gamma}$. Then for the constant curvature space we have $\bar{R}_{\mu\nu} = \Lambda\bar{g}_{\mu\nu}$ so the Lagrangian takes the form:

$$\mathcal{L} = \sqrt{det(\Lambda\bar{g}_{\mu\nu} + \tilde{R}_{\mu\nu})} \tag{16}$$

It is interesting that Lagrangians of such kind have already been investigated e.g. [7, 8, 9, 10, 11, 12] in attempts to construct gravitational analog of the Born-Infeld electrodynamics. But now the interpretation of the Lagrangian is drastically different. Indeed, let us use the well known decomposition for the determinant

$$\sqrt{det(I + A)} = 1 + \frac{1}{2}Sp(A) + \frac{1}{8}(Sp(A))^2 - \frac{1}{4}Sp(A^2) + ...$$

where A is any matrix. Then if we consider the curvature $R_{\mu\nu}$ as being expressed in terms of metric and its second derivatives the first linear terms gives scalar curvature while quadratic terms give higher derivative terms leading to the appearance of ghosts. But here the main dynamical quantity is affine connection Γ and the curvature $R_{\mu\nu}$ contains only first derivatives. As a result a term linear in R is just a total derivative and could be dropped out of the action, while the quadratic terms give exactly the kinetic terms we obtained above.

Finally, let us add some comments on possible interaction with matter in such formulation of gravity. The most clear and straightforward way to obtain these interactions is to start

with usual interactions in first order form and then try to go to the dual formulation. For example, for the scalar field we get:

$$\mathcal{L} = \sqrt{-g}[g^{\mu\nu}R_{\mu\nu} + \Lambda + \frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \frac{m^2}{2}\varphi^2] =$$
$$= \hat{g}^{\mu\nu}(R_{\mu\nu} + \frac{1}{2}\partial_\mu\varphi\partial_\nu\varphi) + (\Lambda - \frac{m^2}{2}\varphi^2)det(\hat{g}^{\mu\nu})^{\frac{1}{d-2}} \qquad (17)$$

and the second line shows that the main effect is the replacement of $R_{\mu\nu}$ by $R_{\mu\nu} + \frac{1}{2}\partial_\mu\varphi\partial_\nu\varphi$ (compare [12]). Also if scalar field has nonzero mass the cosmological constant Λ is replaced by field dependent combination $\Lambda - \frac{m^2}{2}\varphi^2$. But for the vector field (even massless) the situation turns out to be much more complicated because even for the minimal interaction:

$$\mathcal{L} = \sqrt{-g}[g^{\mu\nu}R_{\mu\nu} + \Lambda - \frac{1}{4}g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta}] =$$
$$= \hat{g}^{\nu\nu}R_{\mu\nu} + \Lambda det(\hat{g}^{\mu\nu})^{\frac{1}{d-2}} - \frac{1}{4}det(\hat{g}^{\mu\nu})^{-\frac{1}{d-2}}\hat{g}^{\mu\alpha}\hat{g}^{\nu\alpha}F_{\mu\nu}F_{\alpha\beta} \qquad (18)$$

equations for the $\hat{g}$ become highly nonlinear. But in a weak field approximation such model could reproduce a correct kinetic term for the vector field. Note also that the corrections to the $R_{\mu\nu}$ tensor here start with the terms quadratic in $F_{\mu\nu}$ and there is no term linear in it in contrast with [12].

Thus we have shown that the dualization procedure based on the use of (Anti) de Sitter background space could be applied to the gravity theory in a Palatini formalism and leads to the formulation in terms of affine connection. In this, the final Lagrangian coincides with that of Eddington [3]. A number of interesting question arises, for example, whose related with the gauge symmetries of such formulation, which deserve further study.

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